Chapter 1: Real Numbers Notes for Class 10 Maths

Real Numbers:

R = Real Numbers:
All rational and irrational numbers together form the set of real numbers.
I = Integers:
Numbers like …, -3, -2, -1, 0, 1, 2, 3 … are called integers.
Q = Rational Numbers:
A rational number can be defined as any number that can be expressed in the form p/q, where p and q are integers and $q \neq 0$. Example: 3/2, 0.333…Their decimal expansions are either terminating or non-terminating but repeating.
Q' = Irrational Numbers:
Real numbers that cannot be expressed as p/q, are irrational. Their decimal expansions are non-terminating and non-repeating. Like $\sqrt{2}$, $\sqrt{3}$, $\sqrt{7}$…
N = Natural Numbers:
Counting numbers:
N = { 1,2,3,…}
W = Whole Numbers:
Natural numbers together with zero:
W={0,1,2,3,…}.

Type	Definition	Examples
Even numbers	Of the form 2n	2, 4, 6, 8...
Odd numbers	Of the form 2n − 1	1, 3, 5, 7...
Prime numbers	Exactly 2 factors: 1 and itself	2, 3, 5, 7, 11, 13...
Composite numbers	More than 2 factors	4, 6, 8, 9, 10...

Euclid's Division Lemma:

Given any two positive integers a and b, there exist unique integers q (quotient) and r (remainder) such that:
$a = bq + r$, where $0 \leq r < b$
When you divide 17 by 5, you get:
$17 = 5 \times 3 + 2$
Here, $a = 17$, $b = 5$, $q = 3$, $r = 2$. And indeed, $0 \leq 2 < 5$.

Euclid's Division Algorithm (Finding HCF):
It gives us a step-by-step method to find the HCF (Highest Common Factor) of two positive integers.
To find HCF of two numbers a and b (where $a > b$):
Step 1: Apply Euclid's Division Lemma to a and b: $a = bq + r$
Step 2: If $r = 0$, then HCF = b. Stop.
Step 3: If $r \neq 0$, replace a with b and b with r. Repeat from Step 1.
Keep going until the remainder becomes zero. The divisor at that step is the HCF.
Example 1: Find HCF of 56 and 72
Step 1: $72 = 56 \times 1 + 16$ ($r = 16 \neq 0$, continue)
Step 2: $56 = 16 \times 3 + 8$ ($r = 8 \neq 0$, continue)
Step 3: $16 = 8 \times 2 + 0$ ($r = 0$)
HCF (56, 72) = 8

The Fundamental Theorem of Arithmetic
Statement: Every composite number can be expressed (factorised) as a product of primes, and this factorisation is unique, apart from the order in which the prime factors occur.
Example: $270 = 2 \times 135$
$= 2 \times 3 \times 45$
$= 2 \times 3 \times 3 \times 15$
$= 2 \times 3 \times 3 \times 3 \times 5$
$= 2 \times 3^3 \times 5$
Question: Check whether 15^n can end with the digit zero for any natural number n.
Solution: If 15^n ends in 0, it must be divisible by 10.
$10 = 2 \times 5$, so for any number to end in 0, its prime factorisation must include both 2 and 5.
$15^n = (3 \times 5)^n = 3^n \times 5^n$

The prime factorisation of 15^n contains only 3s and 5s, there is no factor of 2.
By the Fundamental Theorem of Arithmetic, 15^n can never end with the digit 0 for any natural number n.

HCF and LCM Using Prime Factorisation

Express both numbers as products of prime factors.
HCF = product of the smallest power of each common prime factor.
LCM = product of the greatest power of each prime factor in either number.

Example 1: Find HCF and LCM of 120 and 144
Step 1: Prime factorisation
$120 = 2^3 \times 3 \times 5$
$144 = 2^4 \times 3^2$
Step 2: HCF (smallest power of common primes: 2 and 3)
$HCF = 2^3 \times 3^1 = 8 \times 3 = 24$
Step 3: LCM (greatest power of every prime that appears)
$LCM = 2^4 \times 3^2 \times 5 = 16 \times 9 \times 5 = 720$

HCF × LCM = Product of Two Numbers
For any two positive integers a and b:
HCF(a, b) × LCM(a, b) = a × b
Question: If HCF(a, b) = 4 and LCM(a, b) = 48, and a = 12, find b.
$HCF \times LCM = a \times b$
$4 \times 48 = 12 \times b$
$192 = 12b$
$b = 16$

Revisiting Irrational Numbers
An irrational number is a real number that CANNOT be expressed in the form p/q, where p and q are integers and $q \neq 0$. Their decimal expansions are non-terminating and non-repeating.
Examples: $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, π, e, $\sqrt{11}$, $\sqrt[3]{2}$, $\sqrt{2} + \sqrt{3}$
Key theorem (used in proofs):

If a prime number p divides a^2, then p also divides a, where a is a positive integer.

Proof 1: Prove that $\sqrt{2}$ is Irrational

Method: Proof by contradiction.

Assume $\sqrt{2}$ is rational. Then it can be written as:

$\sqrt{2} = p/q$, where p and q are integers, $q \neq 0$, and p/q is in its lowest terms (i.e., HCF(p, q) = 1 and they share no common factors).

Step 1: Squaring both sides:

$2 = p^2/q^2$

$\Rightarrow p^2 = 2q^2$

This tells us p^2 is even (since it equals $2q^2$, which is a multiple of 2).

Step 2: If p^2 is even, then p must be even.

(Reason: If p were odd, p^2 would also be odd. Since p^2 is even, p must be even.)

Let p = 2k for some integer k.

Step 3: Substituting p = 2k into $p^2 = 2q^2$:

$(2k)^2 = 2q^2$

$4k^2 = 2q^2$

$q^2 = 2k^2$

Therefore q^2 is even, so q is also even.

Step 4: But if both p and q are even, they share a common factor of 2.

This contradicts our assumption that HCF(p, q) = 1.

Conclusion: Our assumption was wrong. $\sqrt{2}$ is irrational.

Proof 2: Prove that $\sqrt{3}$ is Irrational

Assume $\sqrt{3} = p/q$ in lowest terms (HCF(p, q) = 1).

Step 1: Squaring: $p^2 = 3q^2$

$\Rightarrow$ 3 divides p^2

$\Rightarrow$ 3 divides p (If a prime number p divides a^2, then p also divides a, where a is a positive integer)

Let p = 3k.

Step 2: $(3k)^2 = 3q^2$

$9k^2 = 3q^2$

$q^2 = 3k^2$

$\Rightarrow$ 3 divides q^2 $\Rightarrow$ 3 divides q

Step 3: Both p and q are divisible by 3. This contradicts HCF(p, q) = 1.
Conclusion: $\sqrt{3}$ is irrational.

Example 1: Prove that $2 + \sqrt{5}$ is irrational.
Assume $2 + \sqrt{5} = r$ (rational).
$\Rightarrow \sqrt{5} = r - 2$
$r - 2$ is rational (difference of two rationals), but $\sqrt{5}$ is irrational which is a contradiction to the fact that $r - 2$ is rational .
Therefore, $2 + \sqrt{5}$ is irrational.

Example 4: Prove that $1/\sqrt{2}$ is irrational.
Assume $1/\sqrt{2} = p/q$ (rational, in lowest terms).
$\Rightarrow \sqrt{2} = q/p$
q/p is rational. But $\sqrt{2}$ is irrational which is a contradiction.
Therefore, $1/\sqrt{2}$ is irrational.

Key Properties: Sum and Product of Rationals and Irrationals

Operation	Result
Rational + Rational	Always Rational
Rational × Rational	Always Rational
Rational + Irrational	Always Irrational
Non-zero Rational × Irrational	Always Irrational
Irrational + Irrational	Can be Rational OR Irrational
Irrational × Irrational	Can be Rational OR Irrational

Decimal Expansion of Real Numbers

A. Terminating Decimals

Every rational number, when divided, gives a decimal that either terminates or repeats. A decimal expansion that ends after a finite number of digits.

A rational number p/q (in standard form) has a terminating decimal expansion if and only if the prime factorisation of q contains only powers of 2 and/or 5 i.e., q is of the form $2^m \times 5^n$, where m, n are non-negative integers.
If q has any prime factor other than 2 or 5, the decimal will be non-terminating and repeating.
Examples:
1/2 = 0.5
3/8 = 0.375

B. Non-Terminating Repeating (Recurring) Decimals

When the denominator has prime factors other than 2 and 5, the decimal goes on forever but repeats a pattern.
Examples:

$1/3 = 0.\overline{3} = 0.333...$
$1/7 = 0.142857142857... = 0.\overline{142857}$
All non-terminating repeating decimals are rational numbers. They can always be expressed as p/q.

C. Non-Terminating Non-Repeating Decimals → Irrational

If a decimal goes on forever and has no repeating pattern, the number is irrational.
$\sqrt{2} = 1.41421356237...$
$\pi = 3.14159265358…$

Here's everything compressed for quick revision.

Concept	Result / Rule
Euclid's Lemma	$a = bq + r$, $0 \le r < b$ (unique q, r)

HCF via algorithm	Keep dividing until r = 0; last divisor = HCF
Fundamental Theorem	Every composite = unique product of primes
HCF (prime factorisation)	Product of smallest powers of common prime factors
LCM (prime factorisation)	Product of greatest powers of all prime factors
HCF × LCM	= Product of two numbers (a × b)
√p is irrational	If p is a prime number
Terminating decimal	Denominator (in lowest terms) = $2^m \times 5^n$ only
Non-terminating repeating	Denominator has prime factor ≠ 2, 5
Non-terminating non-repeating	Irrational number
Rational + Irrational	Always Irrational
Non-zero Rational × Irrational	Always Irrational
Irr + Irr / Irr × Irr	Can be rational or irrational

Important Theorems at a Glance:

Theorem 1: If a prime p divides a^2, then p divides a (where a is a positive integer). (Used in every irrationality proof)
Theorem 2 (Euclid's Lemma): $a = bq + r$, $0 \leq r < b$, for unique integers q and r.
Theorem 3 (Fundamental Theorem of Arithmetic): Every composite number has a unique prime factorisation.
Theorem 4: Let p/q be a rational in standard form. Its decimal expansion is terminating iff $q = 2^m \times 5^n$.

Theorem 5: Let p/q be a rational in standard form. If q has prime factors other than 2 and 5, its decimal expansion is non-terminating repeating.