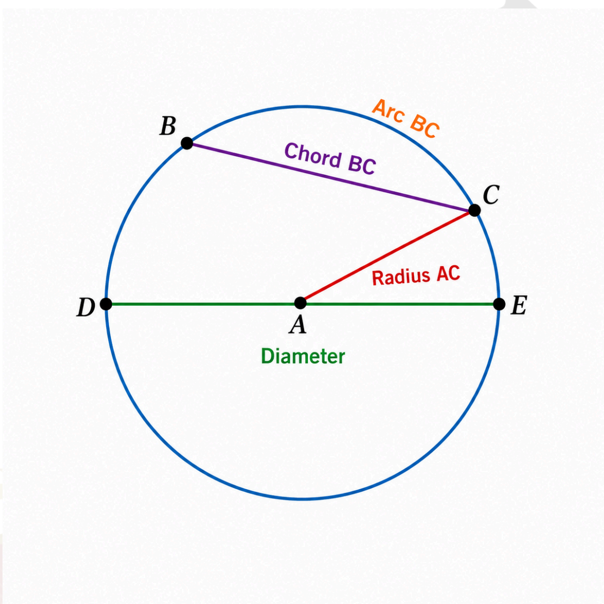
Chapter 5: I'm Up and Down, and Round and Round Class 9 Notes

Key Definitions

- **Circle**: A circle is the set of all points in a plane that are equidistant from a fixed point called the centre. The constant distance is called the radius.
- **Radius**: The distance from the centre to any point on the circle. All radii of the same circle are equal.
- **Chord**: A line segment joining any two points on the circle. Example: if B and C are on the circle, BC is a chord.
- **Diameter**: A chord that passes through the centre. It is the longest chord and equal to twice the radius (d = 2r).
- **Arc**: A connected portion of the circle between two points. The larger part is the major arc; the smaller is the minor arc.
- **Central Angle**: The angle subtended by a chord (or arc) at the centre of the circle.



- **Circumcircle**: The unique circle that passes through all three vertices of a triangle. Its centre is the circumcentre.
- **Cyclic Quadrilateral**: A quadrilateral whose four vertices all lie on a circle. Also called a cyclic 4-gon.
- **Concyclic**: Points that all lie on the same circle are said to be concyclic.

Symmetries of a Circle

There are two types of symmetry in a circle:
1. Rotational symmetry: rotate by any angle about the centre; the circle maps onto itself.
2. Reflection symmetry: every diameter is a line of reflection symmetry. Since there are infinitely many diameters, a circle has infinitely many lines of reflection symmetry.

How Many Circles Can Pass Through Given Points?

Through 1 point: Infinitely many circles (any circle that passes through that point).

Through 2 points A and B: Infinitely many. If a circle passes through A and B, its centre must be equidistant from A and B, meaning the centre lies on the perpendicular bisector of AB. Every single point on that perpendicular bisector is equidistant from A and B, so each such point can serve as the centre of a different circle through A and B.
The smallest circle through two points A and B has its centre at the midpoint of AB (so AB is a diameter), with radius = AB/2.

Through 3 non-collinear points: Exactly ONE circle.

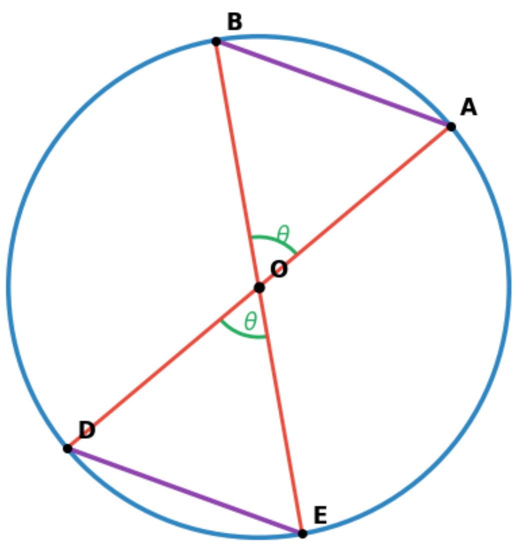
Theorem 1: There is a unique circle passing through three non-collinear points.
Let the three points be A, B, C (not on a straight line). Draw the perpendicular bisectors of AB and AC. Since A, B, C are non-collinear, these two perpendicular bisectors are not parallel, they intersect at exactly one point O. This O is equidistant from all three points (OA = OB = OC), so it's the unique centre of the unique circle through A, B, C.
The circle through the three vertices of a triangle is its circumcircle, and the point O is the circumcentre.

Where does the circumcentre lie?
• Acute-angled triangle: circumcentre is inside the triangle.
• Obtuse-angled triangle: circumcentre is outside the triangle.
• Right-angled triangle: circumcentre lies at the midpoint of the hypotenuse.

What if the 3 points are collinear? No circle can pass through 3 collinear (straight-line) points. The perpendicular bisectors of two segments on the same line are parallel and never meet so there's no common point that could serve as a centre.

Chords and the Angles They Subtend

Theorem 2: Equal chords of a circle subtend equal angles at the centre.



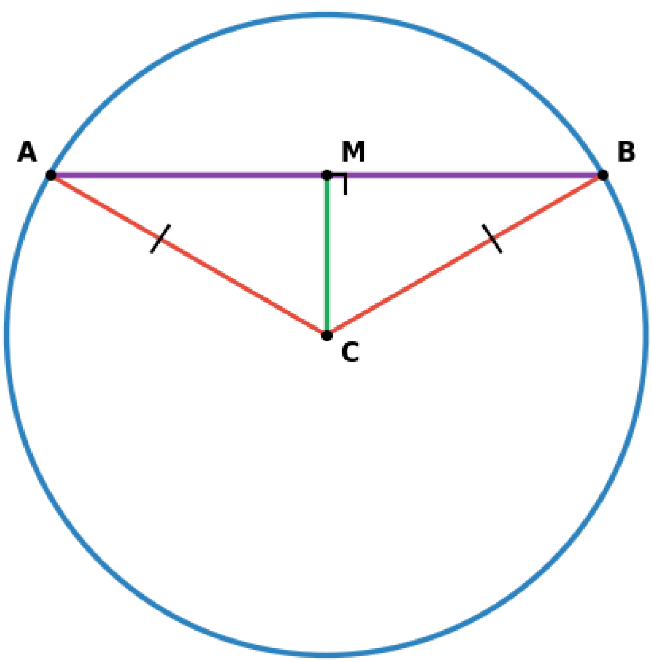
Proof: Let AB and DE be chords with AB = DE.
Since CA = CB = CD = CE = radius (r)
Triangles CAB and CDE have three equal sides (SSS congruence).
Therefore $\angle ACB = \angle DCE$.

Theorem 3 (Converse of Theorem 2)
Chords of a circle that subtend equal angles at the centre are equal.
Proof idea: Given $\angle ACB = \angle DCE$. Since AC = BC = DC = EC = r, by SAS congruence $\triangle ACB \cong \triangle DCE$.

Hence AB = ED.

Midpoints and Perpendicular Bisectors of Chords

Theorem 4: The line joining the centre of a circle and the midpoint of a chord is perpendicular to the chord.

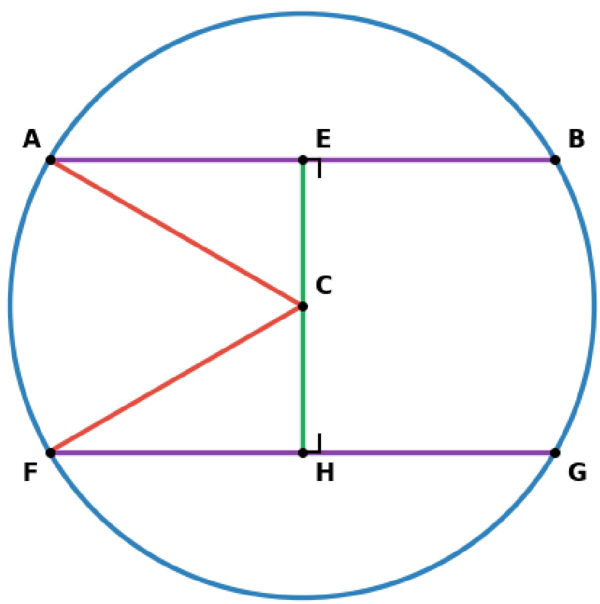


Proof: Let AB be a chord, M its midpoint, C the centre.
$\triangle CAB$ is isosceles (CA = CB = r).
M is the midpoint of AB, so AM = BM.
By SAS congruence, $\triangle CMA \cong \triangle CMB$, giving $\angle CMA = \angle CMB$.
Since these two angles together make 180° (they are on a straight line), each equals 90°. So $CM \perp AB$.

Theorem 5 (Converse of Theorem 4)
The perpendicular from the centre of a circle to a chord bisects the chord.
Proof: Given $CM \perp AB$, then $\angle CMA = \angle CMB = 90°$.
In right triangles CMA and CMB, CM is common and CA = CB = r.
By RHS congruence, AM = BM

Distance of Chords from the Centre

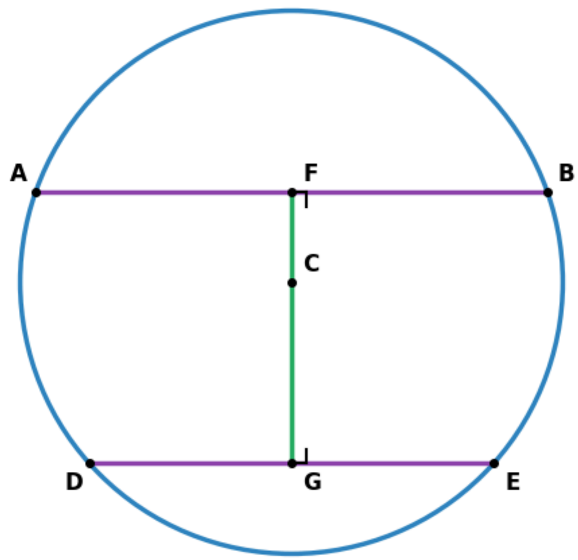
Theorem 6
Chords of a circle having the same length are all at the same distance from the centre.



Proof: Let AB = FG.
Drop perpendiculars CE and CH from centre C to chords AB and FG.
△CEA and △CHF are right triangles with hypotenuse CA = CF (= radius) and equal half-chords.
AE = FH, since AB = FG and E, H are midpoints.
By RHS congruence, CE = CH.

Theorem 7 (Converse of Theorem 6)
Chords of a circle that are equidistant from the centre have equal length.

Theorem 8
If AB and DE are two chords of a circle and AB > DE, then the distance from the centre to AB is less than the distance from the centre to DE. (Longer chord is closer to the centre.)



Proof: Drop perpendiculars CF and CG to AB and DE.
$CF^2 + AF^2 = r^2 = CG^2 + GD^2$.
Since AB > DE, AF > GD (as F and G are midpoints).
So $AF^2 > GD^2$, which gives $CF^2 < CG^2$, i.e., CF < CG.

Note:

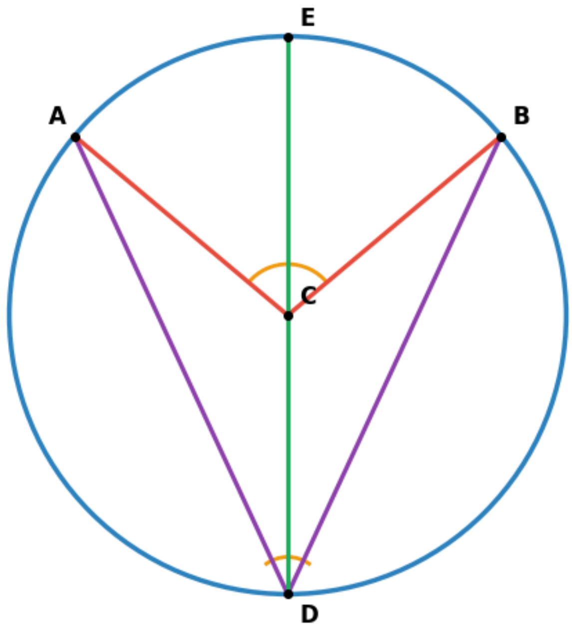
- **Chord Length Formula**
  Chord = $2\sqrt{(r^2 - d^2)}$
  Where
  r = radius, d = perp. distance
- **Distance from Centre**
  $d = \sqrt{(r^2 - (\text{chord}/2)^2)}$

Angles Subtended by an Arc

Two points A and B on a circle divide it into two arcs. The smaller one is the minor arc, and the larger one is the major arc. The central angle subtended by an arc is the angle swept at the centre as we move along that arc.

Theorem 9: The Inscribed Angle Theorem

The angle subtended by an arc at the centre of the circle is double the angle subtended by the same arc at any point on the circle outside the arc.
Proof: Let AFB be an arc. Let C be the centre, D a point outside the arc.
Extend DC to meet the arc at E.



$\triangle$DCB is isosceles (CB = CD), so $\angle$CBD = $\angle$CDB.
By the exterior angle theorem, $\angle$BCE = 2$\angle$BDC.
Similarly, for $\triangle$ADC: $\angle$ACE = 2$\angle$CDA.
Adding: $\angle$BCA = $\angle$BCE + $\angle$ACE = 2$\angle$BDA.

Important Corollary
The angle subtended by a diameter at any point on the circle is always 90°. (Angle in a semicircle is a right angle.)

Concyclicity of Points

Theorem 10
If a line segment AB joining two points A, B subtends equal angles at two other points C, D that lie on the same side of AB, then all four points A, B, C, D lie on a single circle (are concyclic).

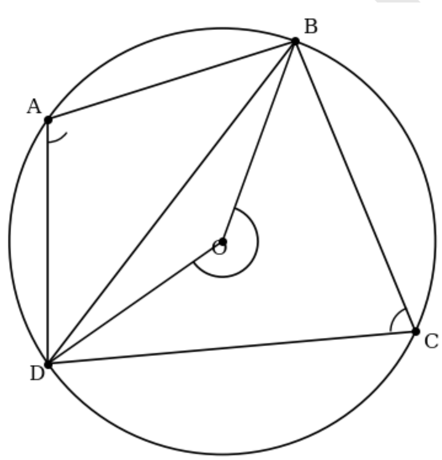
Proof: Suppose A, B, C are non-collinear. By Theorem 1 there is a unique circle through A, B and C. We will prove that D also lies on this circle. Draw the circle through A, B and C. Assume, for contradiction, that D does not lie on the circle; then D is either outside it or inside it. Join A to D.

If D lies outside the circle, the line AD meets the circle again at a point E. If D lies inside the circle, extend AD to meet the circle at E. In either case, the points C and E lie on the same arc determined by the chord AB, so $\angle ACB = \angle AEB$ (angles in the same segment).

Now the two cases:
If D is outside the circle, then $\angle AEB$ is an exterior angle of triangle BED, so $\angle AEB > \angle ADB$. But $\angle AEB = \angle ACB$ (angles in the same segment) and $\angle ACB = \angle ADB$ (given). Thus $\angle ACB > \angle ACB$, a contradiction.

If D is inside the circle, then $\angle ADB$ is an exterior angle of triangle BED, and a similar argument again leads to a contradiction.
Since both possibilities (D outside or inside) produce contradictions, D cannot be outside or inside the circle. Therefore D must lie on the circle through A, B and C.

Theorem 11
The sum of two opposite angles of a cyclic quadrilateral is 180°.



Proof: In cyclic quadrilateral ABCD
Consider arc BCD; point A is on the circle and it lies outside the arc BCD. So $\angle BAD$ is half the angle that arc BCD subtends at the centre O.

So, ∠BAD = 1/2 (reflex angle BOD).
∠BAD = ½ (reflex ∠BOD) and ∠BCD = ½ (∠BOD).
Together: ∠BAD + ∠BCD = ½ × 360° = 180°.

Theorem 12 (Converse of Theorem 11)
If two opposite angles of a quadrilateral add up to 180°, then the four vertices lie on a circle (the quadrilateral is cyclic).