HOTS Questions on Chapter 1 'Real Numbers' for Class 10

Question 1: Prove that the square of any positive integer is of the form 4q or 4q + 1 for some integer q.

Solution:

Step 1: Let n be any positive integer. By Euclid's Lemma with $b = 2$, every integer is either of the form 2m (even) or 2m + 1 (odd).

Case 1 Even: If $n = 2m$, then $n^2 = 4m^2$. Setting $q = m^2$, we get $n^2 = 4q$.

Case 2 Odd: If $n = 2m + 1$, then $n^2 = 4m^2 + 4m + 1 = 4(m^2 + m) + 1$. Setting $q = m^2 + m$, we get $n^2 = 4q + 1$.

Every square is of the form 4q or 4q + 1. This also means a square can never be of the form 4q + 2 or 4q + 3.

Question 2: Show that the cube of any positive integer is of the form 9m, 9m + 1, or 9m + 8.

Solution:

Step 1: Divide any positive integer by 3. The remainder is 0, 1, or 2. So every integer has one of these forms: 3q, 3q + 1, or 3q + 2.

Case 1: $n = 3q \Rightarrow n^3 = 27q^3 = 9(3q^3) = 9m$, where $m = 3q^3$.

Case 2: $n = 3q + 1 \Rightarrow n^3 = 27q^3 + 27q^2 + 9q + 1 = 9(3q^3 + 3q^2 + q) + 1 = 9m + 1$.

Case 3: $n = 3q + 2 \Rightarrow n^3 = 27q^3 + 54q^2 + 36q + 8 = 9(3q^3 + 6q^2 + 4q) + 8 = 9m + 8$.

$\therefore$ Any cube is of the form 9m, 9m + 1, or 9m + 8

Question 3: The prime factorisation of a natural number n is $2^3 \times 3^2 \times 5 \times 7$. How many factors does n have? Is n divisible by 45? Justify.

Solution:

Number of factors: If $n = p_1^a \times p_2^b \times p_3^c$..., total factors = (a+1)(b+1)(c+1)...

So factors of $n = 2^3 \times 3^2 \times 5^1 \times 7^1 = (3+1)(2+1)(1+1)(1+1) = 4 \times 3 \times 2 \times 2 = 48$

Is n divisible by 45?

$45 = 3^2 \times 5$. The factorisation of n contains 3^2 and 5^1.

Since both prime powers of 45 appear in n, yes, n is divisible by 45.

Question 4: Find the largest number that divides 70 and 125, leaving remainders 5 and 8 respectively.

Solution:
Step 1: If a number d divides 70 leaving remainder 5, then d exactly divides 70 − 5 = 65. Similarly, d divides 125 − 8 = 117.
Step 2: We need the HCF of 65 and 117.
$65 = 5 \times 13$ and $117 = 9 \times 13 = 3^2 \times 13$
HCF(65, 117) = 13
Verify: $70 \div 13 = 5$ remainder 5 and $125 \div 13 = 9$ remainder 8
Required number = 13

Question 5: Explain why $7 \times 11 \times 13 + 13$ is a composite number.
Solution:
Factorise: $7 \times 11 \times 13 + 13 = 13 \times (7 \times 11 + 1)$
$= 13 \times (77 + 1) = 13 \times 78$
$= 13 \times 2 \times 39 = 13 \times 2 \times 3 \times 13$
The number = $2 \times 3 \times 13^2$, which has more than two factors. By definition, it is a composite number. It is not prime because it has factors other than 1 and itself.

Question 6: Can two numbers have HCF = 18 and LCM = 380? Give reasons.
Solution:
Key Property: HCF of two numbers always divides their LCM.
Check: Does 18 divide 380?
$380 \div 18 = 21.11...$
No, 18 does not divide 380.
Two numbers with HCF = 18 and LCM = 380 cannot exist. This is a contradiction of the fundamental property that HCF | LCM.

Question 7: Find the smallest 4-digit number that is exactly divisible by 8, 12, and 20.
Solution:
Step 1: Find LCM(8, 12, 20).
$8 = 2^3$, $12 = 2^2 \times 3$, $20 = 2^2 \times 5$
LCM = $2^3 \times 3 \times 5 = 120$
Step 2: Find the smallest 4-digit multiple of 120. Smallest 4-digit number = 1000.
$1000 \div 120 = 8.33...$
The next whole = 9.

$120 \times 9 = 1080$
Smallest 4-digit number = 1080

Question 8: If a and b are rational and p and q are irrational, is $a + b\sqrt{p}$ always irrational? Give a counterexample or prove it.
Solution:
It depends on b: If $b = 0$, then $a + b\sqrt{p} = a$, which is rational. So the statement is not always true.
But if $b \neq 0$: Assume $a + b\sqrt{p} = r$ (rational). Then $\sqrt{p} = (r - a)/b$. Since a, b, r are all rational and $b \neq 0$, the RHS is rational. But $\sqrt{p}$ is irrational, which is a contradiction.
$a + b\sqrt{p}$ is irrational whenever $b \neq 0$

Question 9: The decimal expansion of a number is 0.142857142857... Is this number rational or irrational? If rational, express it in p/q form.
Solution:
Pattern: '142857' repeats every 6 digits, so this is a non-terminating repeating decimal = rational.
Let $x = 0.\overline{142857}$. Then $1000000x = 142857.\overline{142857}$.
Subtracting: $999999x = 142857$, so $x = 142857/999999 = 1/7$.

Question 10: Three bells ring at intervals of 6, 12, and 18 minutes respectively. They all ring together at 8:00 AM. At what time will they next ring together? How many times will the first bell have rung by then (after 8 AM)?
Solution:
Step 1: Find LCM(6, 12, 18).
$6 = 2 \times 3$, $12 = 2^2 \times 3$, $18 = 2 \times 3^2$
LCM $= 2^2 \times 3^2 = 36$ minutes
Step 2: All three bells will ring together again at 8:00 AM + 36 minutes = 8:36 AM.
Step 3: The first bell rings every 6 minutes. In 36 minutes, it rings $36 \div 6 = 6$ times (at 8:06, 8:12, 8:18, 8:24, 8:30, 8:36).